Profit maximization with two inputs

Consider a firm that utilizes two inputs in its production process with the production function

$$f(x_1, x_2) = 12x_1^{1/3}x_2^{1/2}$$

And faces input prices (p_1, p_2) and an output price q.

1. Demonstrate that the production function f is concave, which implies that the firm's profit is also concave.

2. Determine the global maximum of the firm's profit and identify the combination of inputs (x_1, x_2) that result in this maximum profit.

Solution

1. To show that the function is concave, we first calculate its first derivatives. We then find the second derivatives and examine the definiteness of the Hessian matrix.

$$f'x_1 = 4x_1^{-2/3}x_2^{1/2}$$

$$f'x_2 = 6x_1^{1/3}x_2^{-1/2}$$

   The Hessian matrix at (x_1, x_2) is

$$H = \begin{bmatrix} f''_{x_1x_1} & f''_{x_1x_2} \\ f''_{x_2x_1} & f''_{x_2x_2} \end{bmatrix} = \begin{bmatrix} -(8/3)x_1^{-5/3}x_2^{1/2} & 2x_1^{-2/3}x_2^{-1/2} \\ 2x_1^{-2/3}x_2^{-1/2} & -3x_1^{1/3}x_2^{-3/2} \end{bmatrix}$$

   The leading principal minors are

$$D_1 = -(8/3)x_1^{-5/3}x_2^{1/2} < 0$$
$$D_2 = 8x_1^{-4/3}x_2^{-1} - 4x_1^{-4/3}x_2^{-1} = 4x_1^{-4/3}x_2^{-1} > 0$$

   Hence the Hessian is negative definite, so that f is concave. The profit function is:

$$\Pi = f(x_1, x_2)q - x_1p_1 - x_2p_2$$

   Since this is a sum of concave functions (cost is a convex and concave function), the profit function is also concave.

2. The first-order conditions for a maximum of profit are

$$\Pi'_{x_1} = qf'_1(x_1, x_2) - p_1 = 4qx_1^{-2/3}x_2^{1/2} - p_1 = 0$$
$$\Pi'_{x_2} = qf'_2(x_1, x_2) - p_2 = 6qx_1^{1/3}x_2^{-1/2} - p_2 = 0$$

$$4qx_1^{-2/3}x_2^{1/2} = p_1$$
$$6qx_1^{1/3}x_2^{-1/2} = p_2$$

   Dividing the first equation with the second one:

$$\frac{2}{3}\frac{x_2}{x_1} = \frac{p_1}{p_2}$$

   So: $x_2 = \frac{p_1}{p_2}\frac{3}{2}x_1$ Inserting this in the first equation:

$$4qx_1^{-2/3}[\frac{p_1}{p_2}\frac{3}{2}x_1]^{1/2} = p_1$$

$$\frac{4q}{[p_1p_2]^{1/2}}[\frac{3}{2}]^{1/2} = x_1^{1/6}$$

$$13824\frac{q^6}{[p_1p_2]^3} = x_1$$

   Inserting this in $x_2 = \frac{p_1}{p_2}\frac{3}{2}x_1$:

$$x_2 = \frac{p_1}{p_2}\frac{3}{2}13824\frac{q^6}{[p_1p_2]^3}$$

$$x_2 = 20736 \frac{q^6}{p_1^2 p_2^4}$$

Since the objective function is concave, this input combination is the one that globally maximizes the firm's profit. The value of the maximal profit is obtained by substituting these optimal input values into the profit function.